ELSEVIER

Journal of Geometry and Physics 39 (2001) 30–44

JOURNAL OF GEOMETRY AND PHYSICS

A geometric integration of the extended Lee homomorphism

Augustin Banyaga

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

Received 15 September 2000; received in revised form 27 November 2000

Abstract

We give a geometric integration of the extended Lee homomorphism, yielding a homomorphism on the group of automorphisms of a locally conformal symplectic manifold and interpret its kernel as quotient of a group of symplectic diffeomorphisms of a canonically associated symplectic manifold, on which we construct the Calabi invariants in terms of the $c\mathcal{A}$-cohomology. The value of this global Lee homomorphism on an automorphism is the similitude ratio of some lifting on the associated symplectic manifold. Applications to mechanics are given. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 53C12; 53C15

Subj. Class.: Differential geometry

Keywords: Locally conformal symplectic pair; Locally conformal symplectic manifold; Lee form; Extended Lee homomorphism; Conformal invariants; Similitude ratio; $c\mathcal{A}$-cohomology; d^{ω}-cohomology; Generalized symplectification of contact manifolds

1. Introduction and statement of the results

A locally conformal symplectic (LCS) pair on a smooth manifold M of dimension $2n \geq 4$ is a couple (Ω, ω), where Ω is a non-degenerate 2-form, and ω a closed 1-form such that $\mathrm{d}\Omega = -\omega \wedge \Omega$ [13,14]. The form ω is called the Lee form; it is uniquely determined by Ω (see Lemma 2 in Section 2.2).

If $\omega = 0$ in the definition above, Ω is a symplectic form. The class of symplectic manifolds (which are the natural setting of classical mechanics) is much smaller than the class of manifolds carrying LCS structures, as examples in Section 2.1 show. However, in Section 5, we show that we can use these "weaker" structures to do mechanics.

E-mail address: banyaga@math.psu.edu (A. Banyaga).

0393-0440/01/$ – see front matter
PII: S0393-0440(00)00071-1

If λ is a positive smooth function, then

$$d(\lambda\Omega) = -(\omega - d(\ln\lambda)) \wedge (\lambda\Omega), \quad \text{i.e.} \ (\lambda\Omega, (\omega - d(\ln\lambda))) \tag{1}$$

is again an LCS pair.

Two LCS pairs (Ω, ω) and (Ω', ω') are said to be conformally equivalent, and we note $(\Omega, \omega) \approx (\Omega', \omega')$, iff there exists a smooth positive function λ such that $\Omega' = \lambda\Omega$, and $\omega' = \omega - d(\ln\lambda)$.

An equivalence class $\mathcal{S} = [\Omega, \omega]$ of LCS pairs is called an LCS structure on M and the couple $(M, \mathcal{S})$ an LCS manifold. If an LCS pair (Ω, ω) is equivalent to an LCS pair $(\Omega, 0)$, we say that the LCS structure $\mathcal{S} = [\Omega, \omega]$ is a (global) conformal symplectic structure. These structures were introduced by Lee [13], and have been studied extensively by Vaisman [14].

Throughout this paper, we will assume that all manifolds considered are connected. Our first result is the following simple, but crucial remark.

Theorem 1. *Let $(M, \mathcal{S})$ be an LCS manifold, and let an LCS pair $(\Omega, \omega) \in \mathcal{S}$. Let $\pi : \tilde{M} \to M$ be the minimum regular covering of M associated with the 1-form ω. Let $\lambda : \tilde{M} \to \mathbb{R}$ be a positive function on $\tilde{M}$ such that*

$$\pi^*\omega = d(\ln\lambda). \tag{2}$$

Then $\tilde{\Omega} = \lambda(\pi^*\Omega)$ is a symplectic form on $\tilde{M}$ and its conformal class $\tilde{\mathcal{S}} = [\tilde{\Omega}, 0]$ depends only on $\mathcal{S}$, i.e. is independent of the choice of $(\Omega, \omega) \in \mathcal{S}$ and of λ.

It is well known that the group $\mathcal{A}$ of automorphisms of the covering $\tilde{M}$ is equal to the group of periods of ω [8].

Lemma 1. *Let λ be as above, for any $\tau \in \mathcal{A}$, $(\lambda \circ \tau)/\lambda = c_\tau$ is a constant number, independent of the choice of λ and $\tau \mapsto c_\tau$ is a group homomorphism c from $\mathcal{A}$ to the multiplicative group $\mathbb{R}^+$ of positive real numbers.*

Let $\mathrm{Diff}_{\mathcal{S}}(M)$ be the group of automorphisms of an LCS structure $\mathcal{S}$ on a smooth manifold M. It is clear that for any LCS pair $(\Omega, \omega) \in \mathcal{S}$, then $\mathrm{Diff}_{\mathcal{S}}(M)$ is the set of all diffeomorphisms ϕ of M such that $\phi^*\Omega = f_\phi\Omega$, where f_ϕ is a smooth function on M. We will denote by $\mathrm{Diff}_{\mathcal{S}}(M)_0$ the subgroup formed by those $\phi \in \mathrm{Diff}_{\mathcal{S}}(M)$, which are isotopic to the identity through $\mathrm{Diff}_{\mathcal{S}}(M)$.

The Lie algebra $\mathcal{X}_{\mathcal{S}}(M)$ of infinitesimal automorphisms of $\mathcal{S}$, consists of vector fields X on M such that $L_X\Omega = \delta_X\Omega$, where δ_X is a smooth function on M. Here L_X stands for the Lie derivative in the direction X.

A short calculation shows that for such vector field X:

$$d(\omega(X)) = L_X\omega = -d\delta_X.$$

Hence, $\omega(X) + \delta_X$ is a constant $l(X)$, and the correspondence $X \mapsto l(X)$ is a Lie algebra homomorphism called the extended Lee homomorphism [14]. For the convenience of the reader, a proof of these facts is given in Section 2.2.

Our main result is the following theorem.

Theorem 2. *Let $(M, \mathcal{S})$ be a connected LCS manifold, an LCS pair $(\Omega, \omega) \in \mathcal{S}$, $\pi : \tilde{M} \to M$ the covering associated with ω, a function $\lambda : \tilde{M} \to \mathbb{R}$ such that $\pi^*\omega = \mathrm{d}(\ln \lambda)$.*

For each $\phi \in \mathrm{Diff}_{\mathcal{S}}(M)_0$, let $\tilde{\phi} : \tilde{M} \to \tilde{M}$ be a diffeomorphism covering ϕ, i.e. such that $\pi \circ \tilde{\phi} = \phi \circ \pi$, then

$$\frac{\lambda \circ \tilde{\phi}}{\lambda} \cdot (f_\phi \circ \pi) \tag{3}$$

is a non-zero constant $b_{\tilde{\phi}}$, independent of the choice of λ. If $\hat{\phi}$ is another lifting of ϕ, then $b_{\hat{\phi}} = \sigma \cdot b_{\tilde{\phi}}$, where $\sigma \in \Delta = c(\mathcal{A})$.

The correspondence $\phi \mapsto b_{\tilde{\phi}}$ is a well-defined group homomorphism:

$$\mathcal{L} : \mathrm{Diff}_{\mathcal{S}}(M)_0 \to \mathbb{R}^+/\Delta,$$

which does not depend on the choice of $(\Omega, \omega) \in \mathcal{S}$, i.e. is a conformal invariant.

The number $b_{\tilde{\phi}}$ is the similitude ratio of $\tilde{\phi}$, i.e. $\tilde{\phi}^*\tilde{\Omega} = b_{\tilde{\phi}}\tilde{\Omega}$. Hence the Kernel G of $\mathcal{L}$ is a normal subgroup which can be identified with a quotient of a connected subgroup of the group of symplectic diffeomorphisms of $(\tilde{M}, \tilde{\Omega})$.

Let ϕ_t be the local 1-parameter group of diffeomorphisms generated by an infinitesimal automorphism $X \in \mathcal{X}_{\mathcal{S}}(M)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln(b_{\tilde{\phi}_t}))|_{t=0} = l(X). \tag{4}$$

As a consequence if l is surjective, then $\mathcal{L}$ is non-trivial. This in turn implies that $\tilde{\Omega}$ is exact.

2. Complements and examples

We say that Eq. (4) gives an integration of l by the mapping $\tilde{\phi} \mapsto \ln(b_{\tilde{\phi}})$. Let $\phi \in \mathrm{Diff}_{\mathcal{S}}(M)$ and f_ϕ the smooth function such that $\phi^*\Omega = f_\phi\Omega$ for some $(\Omega, \omega) \in \mathcal{S}$. Then $\phi^*\,\mathrm{d}\Omega = -\phi^*\omega \wedge \phi^*\Omega = -\phi^*\omega \wedge f_\phi\Omega = \mathrm{d}\phi^*\Omega = \mathrm{d}(f_\phi\Omega) = \mathrm{d}f_\phi \wedge \Omega - f_\phi\omega \wedge \Omega$. Hence $(\phi^*\omega - \omega) \wedge \Omega = -(\mathrm{d}f_\phi/f_\phi) \wedge \Omega$, which, by Lemma 2, gives

$$(\phi^*\omega - \omega) = -\mathrm{d}(\ln f_\phi). \tag{5}$$

If $u(x)$ is any smooth function such that $(\phi^*\omega - \omega) = \mathrm{d}u$, then (5) implies that

$$K = u + \ln f_\phi \tag{6}$$

is a constant depending on ϕ and the choice of the primitive u. There are several ways to choose such a primitive. A first way is to use an isotopy ϕ_t from the identity to ϕ:

$$u(\phi_t) = \int_0^1 \phi_t^*(\mathrm{i}(\dot{\phi_t})\omega)\,\mathrm{d}t,$$

where $\dot{\phi_t}$ is the family of vector fields along the isotopy ϕ_t. Then

$$K(\phi_t) = u(\phi_t) + \ln(f_\phi)$$

is a constant, depending only on the isotopy class relatively to fixed ends $[\phi]$, of the isotopy ϕ_t. The correspondence $[\phi] \mapsto K(\phi_t)$ is the homomorphism $\tilde{\Phi}$ from the universal covering of $\mathrm{Diff}_{\mathcal{S}}(M)_0$ to $\mathbb{R}$ constructed in [10]. We will show that

$$K(\phi_t) = \ln(b_{\tilde{\phi}}), \tag{7}$$

where $\tilde{\phi} = \tilde{\phi}_1$, and $\tilde{\phi}_t$ is the lift of the isotopy ϕ_t with $\tilde{\phi}_0 = \mathrm{id}$. Hence,

$$(\tilde{\Phi}([\phi]) - \ln(b_{\tilde{\phi}})) \in \Delta.$$

This means that $\ln \circ \mathcal{L}$ coincides with Φ of [10].

The first advantage of our construction is that we make no assumption on supports: our homomorphism is defined even for non-compact manifolds. More importantly, that it is a globalization of the observation by Vaisman [14] that $l(X)$ is the constant such that $L_X \Omega_U = l(X)\Omega_U$, where Ω_U is a symplectic form equal to the restriction of Ω to a contractible open subset U of M multiplied by a positive function λ, with $\omega|_U = \mathrm{d}\ln(\lambda)$ (Section 3).

Finally, the kernel G of $\mathcal{L}$ is isomorphic to $\tilde{G}/\mathcal{A}$, where $\tilde{G}$ is the subgroup of the group of symplectic diffeomorphisms $\mathrm{Diff}_{\tilde{\Omega}}(\tilde{M})_0$ of $(\tilde{M}, \tilde{\Omega})$, generated by lifts $\tilde{\phi}$ of elements of $\mathrm{Diff}_{\mathcal{S}}(M)_0$ such that $b_{\tilde{\phi}} = 1$.

We may now apply to $\tilde{G}$ the techniques, and machineries of [1,3]. For instance, the Calabi invariant [3,6] on $\tilde{G}$ is just the restriction of the Calabi invariant on $\mathrm{Diff}_{\tilde{\Omega}}(\tilde{M})_0$. Explicit formulas are given in Sections 3 and 4. This explains the results of [10] saying that the main results of [1,3] in symplectic geometry hold in the LCS case.

For instance, fragmentation lemmas for elements in the kernel of the Calabi homomorphisms are easily proved (Lemma 3). From there, one can go on to study the situation locally, which is the usual symplectic case, thanks to Vaisman remark and its extension herein.

2.1. Examples

Let (N, α) be a contact manifold. The contact form α is a 1-form such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is everywhere non-zero. Here the dimension of M is $2n+1$. Consider the Cartesian product $M = N \times S^1$, and the projections $p_1 : M \to N$, $p_2 : M \to S^1$. Let β be the canonical 1-form on S^1 with integral 1. If we set $\theta = p_1^*\alpha$ and $\omega = p_2^*\beta$, then $\Omega = \mathrm{d}\theta + \omega \wedge \theta$ is non-degenerate and $\mathrm{d}\Omega = -\omega \wedge \mathrm{d}\theta = -\omega \wedge (\Omega - \omega \wedge \theta) = -\omega \wedge \Omega + \omega \wedge \omega \wedge \theta = -\omega \wedge \Omega$. Hence, (Ω, ω) is an LCS pair on M. Here the manifold $\tilde{M} = N \times \mathbb{R}$, and $(\tilde{M}, \tilde{\Omega})$ is the symplectification of the contact manifold (N, α).

The above example leads to *generalized symplectifications of contact manifolds*. Let (N, α) be a contact manifold and consider a smooth increasing function $f(t)$ on the real line. On $M = N \times \mathbb{R}$, consider the 2-form:

$$\Omega_f = \mathrm{d}\theta + \mathrm{d}(f \circ p_2) \wedge \theta, \tag{8}$$

where p_1, p_2 are the projections on the factors N and $\mathbb{R}$, and $\theta = p_1^*\alpha$.

It is clear that Ω_f is non-degenerate, and a short calculation gives

$$\mathrm{d}\Omega_f = -\mathrm{d}(f \circ p_2) \wedge \Omega_f.$$

Hence $(\Omega_f, \mathrm{d}(f \circ p_2))$ is an LCS structure which is conformally equivalent to the symplectic structure:

$$\tilde{\Omega}_f = \mathrm{e}^{(f \circ p_2)}(\mathrm{d}\theta + \mathrm{d}(f \circ p_2) \wedge \theta) = \mathrm{d}(\mathrm{e}^{(f \circ p_2)}\theta). \tag{9}$$

Thus, we have constructed a new class of symplectic structures on $N \times \mathbb{R}$. In particular, when $f(t) = t$, we get the usual symplectification.

Stein manifolds carry complete Liouville fields [7]. Thus if $\tilde{M}$ is a Stein manifold with a complex structure J compatible with the symplectic form $\tilde{\Omega}$, Liouville vector fields can be obtained from J-convex functions [7] (see also [4]). If we can find such a J-convex function, which is invariant by the deck transformations $\mathcal{A}$, and whose Liouville field is complete, then $\mathcal{L}$ is non-trivial.

2.2. *Some classical facts*

Several formulas in this work are deducted using the following well-known fact. A proof is given for convenience of the reader.

Lemma 2. *If Ω is a non-degenerate 2-form on a smooth manifold M of dimension $2n \geq 4$, and α is a 1-form, then $\alpha \wedge \Omega = 0$ implies that α is identically zero.*

Proof. Suppose there is a point $x \in M$ so that $\alpha_x \neq 0$. Complete α_x into a basis of the cotangent space $T_x^*(M)$: $(\alpha_x, \theta_2, \dots, \theta_m)$, $m = 2n$ and write Ω_x as

$$\Omega_x = \sum_{j=2}^{m} u_j \alpha_x \wedge \theta_j + \sum_{i \lneq j} v_{ij} \theta_i \wedge \theta_j.$$

We have

$$0 = \alpha_x \wedge \Omega_x = \sum_{i \lneq j} v_{ij} \alpha_x \wedge \theta_i \wedge \theta_j.$$

Since $\alpha_x \wedge \theta_i \wedge \theta_j$ are linearly independent, $v_{ij} = 0$ for all i and j. Hence, $\Omega_x = \alpha_x \wedge \beta$ with $\beta = \sum_{j=2}^{m} u_j \theta_j$. Consequently, $\Omega_x^2 = 0$ contradicting the fact that $\Omega_x^n \neq 0$, and $n \geq 2$. □

2.3. *Consequences of Lemma 2*

2.3.1. *Uniqueness of the Lee form*

If $\mathrm{d}\Omega = -\omega \wedge \Omega = -\omega' \wedge \Omega$, then $(\omega - \omega') \wedge \Omega = 0$. Hence $\omega' = \omega$.

2.3.2. *The extended Lee homomorphism*

For $X \in \mathcal{X}_{\mathcal{S}}(M)$, we have $L_X \Omega = \delta_X \Omega$. Hence $\mathrm{d}L_X \Omega = \mathrm{d}\delta_X \wedge \Omega - \delta_X(\omega \wedge \Omega)$, which is equal to $L_X \,\mathrm{d}\Omega = L_X(-\omega \wedge \Omega) = -L_X \omega \wedge \Omega - \omega \wedge (\delta_X \Omega)$. Since $L_X \omega = \mathrm{d}(\omega(X))$, we have $\mathrm{d}(\omega(X)+\delta_X)\wedge\Omega=0$. By Lemma 2, $\mathrm{d}(\omega(X)+\delta_X)=0$. Hence, $\omega(X)+\delta_X$ is a constant $l(X)$.

It is obvious that the map l is a vector space homomorphism. To show it is a Lie algebra homomorphism, we only need to show that it vanishes on brackets of vector fields in $\mathcal{X}_S(M)$. For $X, Y \in \mathcal{X}_S(M)$, we have

$$L_{[X,Y]}\Omega=\delta_{[X,Y]}\Omega=L_XL_Y\Omega-L_YL_X\Omega=(X\cdot\delta_Y-Y\cdot\delta_X)\Omega.$$

Hence, $\delta_{[X,Y]}=(X\cdot\delta_Y-Y\cdot\delta_X)$. On the other hand, since ω is a closed 1-form:

$$\omega([X,Y])=L_X\mathrm{i}(Y)\omega-\mathrm{i}(Y)L_X\omega=X\cdot\omega(Y)-Y\cdot\omega(X).$$

Moreover, for any pair of vector fields $U, V,\ V\in\mathcal{X}_S(M)$, we have

$$U\cdot\delta_V=-U\cdot\omega(V),$$

since $\delta_V=-\omega(V)+l(V)$ and $l(V)$ is constant. Therefore,

$$l([X,Y])=\delta_{[X,Y]}+\omega([X,Y])=(X\cdot\delta_Y-Y\cdot\delta_X)+(X\cdot\omega(Y)-Y\cdot\omega(X))=0.$$

2.3.3. A classical result of P. Libermann [12]

Theorem 3. *Let (M,Ω) be a connected symplectic manifold of dimension greater than or equal to* 4. *If ϕ is a diffeomorphism of M such that $\phi^*\Omega=f\Omega$ for some smooth function f, then f is a constant.*

Proof. $0=\phi^*(\mathrm{d}\Omega)=\mathrm{d}(\phi^*\Omega)=\mathrm{d}f\wedge\Omega$. By Lemma 2, $\mathrm{d}f=0$. □

Therefore, if $\mathcal{S}$ is a global conformal symplectic structure on a smooth manifold M, represented by a symplectic form Ω, then $\mathrm{Diff}_{\mathcal{S}}(M)$ consists of those diffeomorphisms h such that $h^*\Omega=k\Omega$, for some constant k, provided that the dimension of M is greater than or equal to 4.

3. The associated conformal symplectic structure

Let (Ω,ω) be an LCS pair on a smooth manifold M and let $\mathcal{U}=(U_i)$ be an open cover by contractible open sets such that $U_i\cap U_j$ is connected. Let ω_i and Ω_i denote, respectively, the restrictions of ω and Ω to the open set U_i. By Poincare lemma, there exist smooth positive functions λ_i on U_i such that $\omega_i=\mathrm{d}(\ln\lambda_i)$. Then $\Omega_i'=\lambda_i\Omega_i$ is non-degenerate and closed, i.e. it is a symplectic form on U_i. Indeed

$$\mathrm{d}(\lambda_i\Omega_i)=\mathrm{d}\lambda_i\wedge\Omega_i+\lambda_i(-\omega\wedge\Omega)=\mathrm{d}\lambda_i-\lambda_i\,\mathrm{d}(\ln\lambda_i)\wedge\Omega_i=0. \tag{10}$$

On $U_i\cap U_j$, we have $\Omega_i'=c_{ij}\Omega_j'$ with $c_{ij}=\lambda_i/\lambda_j$, and $0=\mathrm{d}(\ln\lambda_i)-\mathrm{d}(\ln\lambda_j)=\mathrm{d}(\ln(\lambda_i/\lambda_j))$. Since $U_i\cap U_j$ is connected, $\ln(\lambda_i/\lambda_j)$ and hence $c_{ij}=\lambda_i/\lambda_j$ is a constant.

We thus see that an LCS structure is a Γ-structure on M, where Γ is the pseudogroup of local diffeomorphisms of $\mathbb{R}^{2n}$ preserving the standard symplectic form of $\mathbb{R}^{2n}$ up to a constant number.

We have the following *Darboux*-type theorem.

Theorem 4. *Each point in smooth manifold equipped with an LCS pair has an open neighborhood U and local coordinates* $(x_1, \dots, x_n, y_1, \dots, y_n)$, with $y_1 \neq 0$, such that

$$\Omega_{|U} = y_1 \left(\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \right),$$

and $\omega_{|U} = \mathrm{d}y_1/y_1$.

Vaisman observation [14]. Let $X \in \mathcal{X}_{\mathcal{S}}(M)$ and $\delta(X)$ be the function such that $L_X\Omega = \delta(X)\Omega$. On each U_i, we have

$$L_X\Omega_i' = l(X)\Omega_i'.$$

Theorem 1 simply expresses that the local conformal symplectic structures on U_i's fit together when lifted to an appropriate cover, and Theorem 2 says that l globalizes into a homomorphism whose value on an global automorphism is the similitude ratio of some lifting to the symplectic manifold in Theorem 1.

Proof of Theorem 1. Since π is a local diffeomorphism, $\pi^*\Omega$ is non-degenerate, and so is $\tilde{\Omega} = \lambda\pi^*\Omega$ since λ is a nowhere zero function. The same calculation as in (10) shows that $\mathrm{d}(\lambda\tilde{\Omega}) = 0$. Hence, $\tilde{\Omega}$ is a symplectic form on $\tilde{M}$.

If λ' is so that $\mathrm{d}(\ln\lambda') = \mathrm{d}(\ln\lambda) = \pi^*\omega$, then $\lambda' = c \cdot \lambda$ for some constant c. Hence, $\tilde{\Omega}' = \lambda'\pi^*\Omega' = c \cdot \tilde{\Omega}$. The conformal class $\tilde{\mathcal{S}}$ of $[\tilde{\Omega}, 0]$ is thus independent of the choice of λ.

Let $(\Omega', \omega') \in \mathcal{S}$ be another representative of $\mathcal{S}$. We know that $\Omega' = \mu\Omega$ and $\omega' = \omega - \mathrm{d}(\ln\mu)$ for some positive function μ.

The covering associated with ω' is the same as the one associated with ω, since both coverings have the group of periods of ω as deck transformations [8]. We have $\pi^*\omega' = \mathrm{d}(\ln(\lambda/(\mu\circ\pi)))$. Hence,

$$\tilde{\Omega}' = \left(\frac{\lambda}{\mu\circ\pi}\right)\pi^*\Omega' = \left(\frac{\lambda}{\mu\circ\pi}\right)(\mu\circ\pi)\pi^*\Omega = \tilde{\Omega}.$$

Hence the global conformal symplectic structure $\tilde{\mathcal{S}}$ on $\tilde{M}$ is uniquely determined by $\mathcal{S}$. □

Proof of Lemma 1. Clearly if $\lambda' = a\lambda$ for some constant a, then $(\lambda'\circ\tau)/\lambda' = (\lambda\circ\tau)/\lambda$.

For any $\tau \in \mathcal{A}$, we have

$$\mathrm{d}(\ln(\lambda\circ\tau) - \ln\lambda) = \tau^*\pi^*\omega - \pi^*\omega = (\pi\tau)^*\omega - \pi^*\omega = \pi^*\omega - \pi^*\omega = 0.$$

Hence $\ln((\lambda\circ\tau)/\lambda) = K$ a constant and then $(\lambda\circ\tau)/\lambda = \mathrm{e}^K = c_\tau$.

If $\tau, \tau' \in \mathcal{A}$, then

$$c_{\tau\tau'} = \frac{\lambda \circ \tau\tau'}{\lambda} = \left(\frac{\lambda \circ (\tau\tau')}{\lambda \circ \tau'}\right) \cdot \frac{\lambda \circ \tau'}{\lambda}$$

$$= \left(\left(\frac{\lambda \circ \tau}{\lambda}\right) \circ \tau'\right) \cdot \left(\frac{\lambda \circ \tau'}{\lambda}\right) = \left(\frac{\lambda \circ \tau}{\lambda}\right) \cdot \left(\frac{\lambda \circ \tau'}{\lambda}\right) = c_\tau \cdot c_{\tau'}. \qquad \square$$

Vaisman observation extends into the following proposition.

Proposition 1. *Let $X \in \mathcal{X}_{\mathcal{S}}(M)$ and $\tilde{X}$ a lift of X to the cover $\tilde{M}$*, then

$$L_{\tilde{X}}\tilde{\Omega} = l(X)\tilde{\Omega}.$$

Proof.

$$L_{\tilde{X}}\tilde{\Omega} = ((\tilde{X}) \cdot \lambda)\pi^*\Omega + \lambda L_{\tilde{X}}\pi^*\Omega = \left(\frac{\mathrm{d}\lambda}{\lambda}\right)(\tilde{X})\tilde{\Omega} + \lambda\pi^* L_X\Omega$$

$$= (\mathrm{d}\ln(\tilde{X}) + \delta(X) \circ \pi)\tilde{\Omega} = (\pi^*(\omega(X) + \delta(X)))\tilde{\Omega} = l(X)\tilde{\Omega}. \qquad \square$$

3.1. The $c\mathcal{A}$-cohomology and the d^ω*-cohomology*

The set $\mathcal{F}^*_{c\mathcal{A}}(M)$ of all differential forms α on $\tilde{M}$ such that $\tau^*\alpha = c_\tau\alpha$ for all $\tau \in \mathcal{A}$ is a subcomplex of the de Rham complex of $\tilde{M}$. We denote its cohomology by $H^*_{c\mathcal{A}}(M)$ and call it the conformally $\mathcal{A}$-invariant cohomology of $\tilde{M}$. Clearly, if the cohomology class of ω is trivial, then $H^*_{c\mathcal{A}}(M)$ coincides with the de Rham cohomology of M.

Let us note the following feature of the conformally $\mathcal{A}$-invariant cohomology.

Proposition 2. *Suppose $\Delta \neq \{1\}$, then $H^0_{c\mathcal{A}}(M) = 0$ and for any closed 1-form α representing zero in $H^1_{c\mathcal{A}}(M)$, there is a unique function u such that $\alpha = \mathrm{d}u$ and $u \circ \tau = c_\tau u\ \forall\tau \in \mathcal{A}$.*

Proof. An element of $H^0_{c\mathcal{A}}(M)$ is represented by a constant K such that $K \circ \tau = K = c_\tau K$ for all $\tau \in \mathcal{A}$. Hence, if $\Delta \neq \{1\}$, $K = 0$.

Also if $\alpha = \mathrm{d}u = \mathrm{d}u'$, with $u \circ \tau = c_\tau u$, and $u' \circ \tau = c_\tau u'$, then $u' - u$ is a constant K such that $K = K \circ \tau = c_\tau K$. As above, $K = 0$. $\square$

Proposition 3. *Let Kerl denote the kernel of l which is a Lie algebra of vector fields X such that $L_{\tilde{X}}\tilde{\Omega} = 0$ (i.e. symplectic vector fields), the mapping*

$$s_\lambda : \mathrm{Ker}l \to H^1_{c\mathcal{A}}(M),$$

which assigns to X the cohomology class $[\lambda\pi^*(\mathrm{i}(X)\Omega)]$, is a surjective Lie algebra homomorphism.

This is the equivalent of the Calabi homomorphism [1,3,6].

Proof. By definition $\alpha = \mathrm{i}(\tilde{X})\tilde{\Omega} = \lambda\pi^*(\mathrm{i}(X)\Omega)$ is a closed form. Moreover, we have

$$\tau^*\alpha = (\lambda \circ \tau) \cdot \tau^*\pi^*(\mathrm{i}(X)\Omega) = (\lambda \circ \tau) \cdot \pi^*(\mathrm{i}(X)\Omega) = \left(\frac{\lambda \circ \tau}{\lambda}\right)\alpha = c_\tau \alpha.$$

Hence s_λ maps Kerl into $H^1_{c\mathcal{A}}(M)$. The mapping is a Lie algebra homomorphism since for any $X, Y \in \operatorname{Ker}\mathcal{L}$ and lifts $\tilde{X}, \tilde{Y}$, then $\mathrm{i}([\tilde{X}, \tilde{Y}])\tilde{\Omega} = \mathrm{d}u$ where $u = \tilde{\Omega}(\tilde{X}, \tilde{Y}) = \lambda\Omega(X, Y) \circ \pi$. This implies that $u \circ \tau = c_\tau u\ \forall \tau \in \mathcal{A}$. Hence, $s_\lambda([X, Y]) = 0$.

Let β be a closed 1-form representing an element of $H^1_{c\mathcal{A}}(M)$, and let $\tilde{X}$ be a vector field on $\tilde{M}$ such that $\mathrm{i}(\tilde{X})\pi^*\Omega = \beta/\lambda$. Then for any $\tau \in \mathcal{A}$, we have

$$\begin{aligned}\tau^*(\mathrm{i}(\tilde{X})\pi^*\Omega) &= \mathrm{i}((\tau)_*^{-1}\tilde{X})\tau^*\pi^*\Omega = \mathrm{i}((\tau)_*^{-1}\tilde{X})\pi^*\Omega = \frac{\tau^*\beta}{\lambda} \circ \tau \\ &= \left(\frac{c_\tau \cdot \beta}{\lambda}\right) \cdot \left(\frac{\lambda}{\lambda \circ \tau}\right) = \frac{\beta}{\lambda} = \mathrm{i}(\tilde{X})\pi^*\Omega.\end{aligned}$$

Hence, $\mathrm{i}((\tau)_*^{-1}\tilde{X})\pi^*\Omega = \mathrm{i}(\tilde{X})\pi^*\Omega$, which implies that $(\tau)_*^{-1}\tilde{X} = \tilde{X}$. Therefore, $\tilde{X}$ is the lift of a vector field X on M, and $s_\lambda(X) = \beta$. □

Remark. If $X \in \operatorname{Ker} s_\lambda$ and u is the unique function such that $\lambda\pi^*(\mathrm{i}(X)\Omega) = \mathrm{d}u$, $u \circ \tau = c_\tau u$, then $v = u/\lambda$ is a basic function. If M is compact, we may integrate this function with the measure Ω^n ($2n$ being the dimension of M), and get a number

$$\rho(X) = \int_M v\Omega^n,$$

which obviously depends linearly on X. Is it a Lie algebra homomorphism? Clearly, if $X, Y \in \operatorname{Ker} l$, then

$$\rho([X, Y]) = \int_M \Omega(X, Y)\Omega^n.$$

Lemma 3 (Fragmentation). *Given $X \in \operatorname{Ker} s_\lambda$ and an open cover $\mathcal{U} = (U_i)$ of M, then $X = \sum X_j$ where $X_i \in \mathcal{X}_\mathcal{S}(M)$ has support in U_i.*

Proof. A vector field X is in the kernel of s_λ iff $\lambda(\pi^*\mathrm{i}(X)\Omega) = \mathrm{d}u$, with $u \circ \tau = c_\tau u$ for all $\tau \in \mathcal{A}$. Let (μ_j) be a partition of unity subordinate to $\mathcal{U}$, define a vector fields Y_j on $\tilde{M}$ by $\mathrm{i}(Y_j)\pi^*\Omega = \mathrm{d}((\mu_j \circ \pi)u)/\lambda$. The same arguments as above show that Y_j is basic, i.e. there exists X_j on M with $\pi_* Y_j = X_j$. Then $X = \sum_j X_j$, $\operatorname{supp}(X_j) \subset U_j$ and $X_j \in \mathcal{X}_\mathcal{S}(M)$. □

Remarks.

1. The homomorphism s_λ below depends on λ: if λ' is another function with $\pi^*\omega = \mathrm{d}(\ln \ln \lambda')$, then $\lambda' = k\lambda$, $k \in \mathbb{R}$. Hence $s_{\lambda'} = ks_\lambda$. Hence s_λ gives a well-defined homomorphism into $H^1_{c\mathcal{A}}(M)/\mathbb{R}$. However, the kernel of s_λ is independent of the choice of λ.

2. This homomorphism and other Calabi homomorphisms [1,3,6] were written by Haller–Rybicki [10] using the d^ω-cohomology $H^\omega(M)$ of a manifold M, equipped with an LCS (Ω, ω), introduced by Lichnerowicz. This is the cohomology of the complex of differential forms on M with the differential

$$d^\omega(\alpha) = d\alpha + \omega \wedge \alpha. \tag{11}$$

This cohomology is not an invariant of the LCS structure $\mathcal{S} = [\Omega, \omega]$, although, given $(\Omega', \omega') \in \mathcal{S}$, there is an isomorphism between $H^\omega(M)$ and $H^{\omega'}(M)$, depending on the choice of λ such that $\omega' = \omega - d\ln\lambda$. More precisely, the isomorphism is given by $\alpha \mapsto \lambda\alpha$. The link between these two cohomologies is given by the following lemma.

Lemma 4. *For any differential form*, α, $d^\omega\alpha = 0$ *if and only if* $d(\lambda\pi^*\alpha) = 0$.

Proof. Suppose $d^\omega\alpha = 0$. Then $d(\lambda\pi^*\alpha) = d\lambda \wedge \pi^*\alpha + \lambda\pi^*(-\omega \wedge \alpha) = d\lambda \wedge \pi^*\alpha - \lambda\, d(\ln\lambda) \wedge \pi^*\alpha = 0$.

Suppose now $d(\lambda\pi^*\alpha) = 0$, and compute

$$\lambda\pi^*(d^\omega\alpha) = \lambda\pi^*\, d\alpha + \lambda\pi^*\omega \wedge \pi^*\alpha = \lambda\pi^*\, d\alpha + \lambda\, d(\ln\lambda) \wedge \pi^*\alpha = d(\lambda\pi^*\alpha) = 0.$$

Since λ is a positive function and π is a local diffeomorphism, $d^\omega\alpha = 0$. □

Proposition 4. $H^*_{c\mathcal{A}}(M)$ is a quotient of $H^\omega(M)$.

Proof. The natural homomorphism

$$H^\omega(M) \to H^*_{c\mathcal{A}}(M),$$

$[\alpha] \mapsto [\lambda\pi^*\alpha]$ which admits a section: let β be a form such that $d\beta = 0$ and $\tau^*\beta = c_\tau\beta$ for all $\tau \in \mathcal{A}$. Then

$$\tau^*\left(\frac{\beta}{\lambda}\right) = \frac{\tau^*\beta}{\lambda \circ \tau} = \left(\frac{c_\tau \cdot \beta}{\lambda}\right) \cdot \left(\frac{\lambda}{\lambda \circ \tau}\right) = \frac{\beta}{\lambda}$$

for all $\tau \in \mathcal{A}$. Hence β/λ is basic, i.e. there is a form α on M such that $\beta/\lambda = \pi^*\alpha$. Since $\beta = \lambda\pi^*\alpha$ is closed, α is d^ω closed, by Lemma 4. □

Remark. The manifold $\tilde{M}$ is never compact, unless M is compact and ω is exact, in which case $\tilde{M} = M$. For instance, we note this stronger result.

Suppose the LCS pair (Ω, ω) on M is such that $\Omega = d\theta + \omega \wedge \theta$ for some 1-form θ, then $\tilde{\Omega}$ is exact. Indeed, denoting $\hat{\theta} = \pi^*\theta,\ \hat{\Omega} = \pi^*\Omega,\ \hat{\omega} = \pi^*\omega$, we have

$$\tilde{\Omega} = \lambda\hat{\Omega} = \lambda\, d\hat{\theta} + \lambda\hat{\omega} \wedge \hat{\theta} = \lambda\, d\hat{\theta} + \lambda\, d\ln\lambda \wedge \hat{\theta} = \lambda\, d\hat{\theta} + d\lambda \wedge \hat{\theta} = d(\lambda\hat{\theta}).$$

This implies that $\tilde{M}$ is not compact. Indeed, no compact manifold can carry an exact symplectic form, since the cohomology class of its maximum power is non-zero by Stokes theorem.

4. Proof of Theorem 2

Since π is a covering map, any isotopy $\phi_t : M \to M$ lifts to an isotopy $\tilde{\phi}_t : \tilde{M} \to \tilde{M}$. Hence, $\phi \in \mathrm{Diff}_{\mathcal{S}}(M)_0$ lifts to a diffeomorphism $\tilde{\phi}$ of $\tilde{M}$. This diffeomorphism belongs to the group $\mathrm{Diff}_{\tilde{\mathcal{S}}}(\tilde{M})_0$ of automorphisms of the global conformal symplectic structure $\tilde{\mathcal{S}}$. Indeed

$$\begin{aligned}\tilde{\phi}^*\tilde{\Omega} &= \tilde{\phi}^*(\lambda\pi^*\Omega) = (\lambda \circ \tilde{\phi}) \cdot (\pi \circ \tilde{\phi})^*\Omega = (\lambda \circ \tilde{\phi}) \cdot (\phi \circ \pi)^*\Omega \\ &= (\lambda \circ \tilde{\phi}) \cdot \pi^*(\phi^*\Omega) = (\lambda \circ \tilde{\phi}) \cdot \pi^*(f_\phi \Omega) \\ &= (\lambda \circ \tilde{\phi}) \cdot (f_\phi \circ \pi)\left(\frac{1}{\lambda}\right)(\lambda\pi^*(\Omega)) = \left(\frac{\lambda \circ \tilde{\phi}}{\lambda}\right) \cdot (f_\phi \circ \pi)\tilde{\Omega}.\end{aligned}$$

Here f_ϕ is the function such that $\phi^*\Omega = f_\phi \Omega$.

Since $\tilde{\Omega}$ is a symplectic form, and $\tilde{\phi}^*\tilde{\Omega} = ((\lambda \circ \tilde{\phi})/\lambda) \cdot (f_\phi \circ \pi)\tilde{\Omega}$,

$$\left(\frac{\lambda \circ \tilde{\phi}}{\lambda}\right) \cdot f_\phi \circ \pi$$

is a constant non-zero number, we denote $b_{\tilde{\phi}}$, by the classical Theorem 3 (the dimension of the manifold is ≥ 4).

Changing λ into $\lambda' = k\lambda$, k a constant, does not change $b_{\tilde{\phi}}$.

If $\hat{\phi}$ is another lifting of ϕ, then

$$\frac{b_{\tilde{\phi}}}{b_{\hat{\phi}}} = \frac{\lambda \circ \tilde{\phi}}{\lambda \circ \hat{\phi}} = \left(\frac{\lambda \circ (\tilde{\phi}\hat{\phi}^{-1})}{\lambda}\right) \circ \hat{\phi}.$$

But $\tau = \tilde{\phi}\hat{\phi}^{-1}$ is an automorphism of the covering. Hence, $(\lambda \circ \tau)/\lambda = c_\tau$ is a constant by Lemma 1. Therefore, $b_{\tilde{\phi}} = c_\tau \cdot b_{\hat{\phi}}$.

The correspondence $\phi \mapsto b_{\tilde{\phi}}$ is a well-defined map

$$\mathcal{L} : \mathrm{Diff}_{\mathcal{S}}(M)_0 \to \mathbb{R}^+/\Delta.$$

Let $\phi, \psi \in \mathrm{Diff}_{\mathcal{S}}(M)_0$ and $\tilde{\phi}, \tilde{\psi}$ their (arbitrary) liftings, then $\tilde{\phi}\tilde{\psi}$ is a lifting of $\phi\psi$. We have

$$\begin{aligned}\mathcal{L}(\phi\psi) &= \left(\frac{\lambda \circ \tilde{\phi}\tilde{\psi}}{\lambda}\right) \cdot (f_{\phi\psi} \circ \pi) = \left(\frac{\lambda \circ \tilde{\phi} \circ \tilde{\psi}}{\lambda}\right) \cdot \left(\frac{\lambda \circ \tilde{\psi}}{\lambda \circ \tilde{\psi}}\right)(f_\phi \circ \psi \cdot f_\psi) \circ \pi \\ &= \left(\frac{\lambda \circ \tilde{\phi}\tilde{\psi}}{\lambda \circ \tilde{\psi}}\right) \cdot (f_\phi \circ \psi \circ \pi) \cdot \left(\frac{\lambda \circ \tilde{\psi}}{\lambda}\right) \cdot (f_\psi \circ \pi) \\ &= \left(\frac{\lambda \circ \tilde{\phi}\tilde{\psi}}{\lambda \circ \tilde{\psi}}\right) \cdot (f_\phi \circ \pi \circ \tilde{\psi}) \cdot \left(\frac{\lambda \circ \tilde{\psi}}{\lambda}\right) \cdot (f_\psi \circ \pi) \\ &= \left(\left[\frac{\lambda \circ \tilde{\phi}}{\lambda} \cdot (f_\phi \circ \pi)\right] \circ \tilde{\psi}\right) \cdot \left(\frac{\lambda \circ \tilde{\psi}}{\lambda}\right) \cdot (f_\psi \circ \pi) = \mathcal{L}(\phi) \cdot \mathcal{L}(\psi).\end{aligned}$$

This shows that our map $\mathcal{L}$ is a group homomorphism.

Let us now prove that it is a conformal invariant: if $(\Omega', \omega') \in \mathcal{S}$ is another representative of $\mathcal{S}$, we have $\Omega' = \mu\Omega$, $\omega' = \omega - \mathrm{d}(\ln\mu)$, $\pi^*\omega = \mathrm{d}(\ln\lambda)$, $\pi^*\omega' = \mathrm{d}(\ln\lambda')$ with $\lambda' = \lambda/(\mu\circ\pi)$. If $\phi \in \mathrm{Diff}_{\mathcal{S}}(M)_0$ and $\phi^*\Omega = f_\phi$, $\phi^*\Omega' = f'_\phi\Omega'$, then $f'_\phi = ((\mu\circ\phi)/\mu)\cdot f_\phi$. An immediate calculation of $\mathcal{L}(\phi)$ using λ' and f'_ϕ yield the same thing as using λ and f_ϕ.

By the very definition of $\mathcal{L}$, its kernel G is made of elements ϕ possessing a lift $\tilde{\phi}$ such that $\tilde{\phi}^*\tilde{\Omega} = \tilde{\Omega}$, i.e. symplectic diffeomorphisms of $(\tilde{M}, \tilde{\Omega})$. With notations of Section 2, $G = \tilde{G}/\mathcal{A}$.

The last assertion follows from in the construction of $\mathcal{L}$: each lift $\tilde{\phi}$ is isotopic to the identity: the lift of the isotopy $\phi_t \in \mathrm{Diff}_{\mathcal{S}}(M)_0$. A result of Iglesias [11] then asserts that if $\tilde{\Omega}$ is not exact, then the similitude ratio of $\tilde{\phi}$ is the same as the one for the identity, i.e. 1. In other words, if $\mathcal{L}$ is not trivial, then $\tilde{\Omega}$ is exact. But Eq. (4) implies that if l is non-trivial, so is $\mathcal{L}$.

Finally, let us prove Eq. (4) clarifying the relation of $\mathcal{L}$ and the infinitesimal extended Lee homomorphism l.

Let ϕ_t be the local 1-parameter group generated by a vector field $X \in \mathcal{X}_{\mathcal{S}}(M)$. Let $\delta(X)$, u_{ϕ_t} be the functions such that $L_X\Omega = \delta(X)\Omega$, and $\phi_t^*\Omega = u_t\Omega$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln(u_{\phi_t}))|_{t=0} = \delta(X).$$

On the other hand,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\ln(\lambda\circ\tilde{\phi}_t) - \ln(\lambda)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \tilde{\phi}_s^*(\mathrm{i}(\dot{\tilde{\phi}}_s^*)(\mathrm{d}\ln\lambda))\,\mathrm{d}s\right)\\ &= \tilde{\phi}_t^*\mathrm{i}(\dot{\tilde{\phi}}_t)\pi^*\omega = \tilde{\phi}_t^*\pi^*\mathrm{i}(\pi_*\dot{\tilde{\phi}}_t)\omega = \pi^*(\phi_t^*\mathrm{i}(\dot{\phi}_t)\omega).\end{aligned}$$

This expression evaluated at $t = 0$ gives $\omega(X)\circ\pi$. We thus have proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln(b_{\phi_t}))|_{t=0} = (\omega(X) + \delta(X))\circ\pi = l(X).$$

Hence, $\mathcal{L}' = \ln\circ\mathcal{L}$ is an integral of l.

If $\tilde{\phi} = \tilde{\phi}_1$ is the end of the isotopy $\tilde{\phi}_t$, lifting ϕ_t, with $\tilde{\phi}_0 = \mathrm{id}$, then

$$\begin{aligned}\ln\left(\frac{\lambda\circ\tilde{\phi}}{\lambda}\right) &= \ln(\lambda\circ\tilde{\phi}) - \ln\lambda = \int_0^1 \tilde{\phi}_s^*(\mathrm{i}(\dot{\tilde{\phi}}_s)(\mathrm{d}\ln\lambda)\,\mathrm{d}s) = \int_0^1 \tilde{\phi}_s^*(\mathrm{i}(\dot{\tilde{\phi}}_s)(\pi^*\omega)\,\mathrm{d}s)\\ &= \int_0^1 \pi^*\phi_s^*(\mathrm{i}(\dot{\tilde{\phi}}_s)(\omega)\,\mathrm{d}s) = \left(\int_0^1 \phi_s^*(\mathrm{i}(\dot{\tilde{\phi}}_s)(\omega)\,\mathrm{d}s)\right)\circ\pi = u\circ\pi.\end{aligned}$$

This proves formula (7) in Section 2. The proof of Theorem 2 is now complete.

Remarks. Both formulas given in [10] and the formula given here contain the expression

$$D_{\phi^{-1}} = \ln(f_\phi).$$

It has been proved in [2] (see also [4]) that the correspondence

$$\phi \mapsto D_{\phi^{-1}}$$

is a 1-cocycle on $\mathrm{Diff}_{\mathcal{S}}(M)_0$ with values in the space $C^\infty(M)$ of smooth functions on M, whose cohomology class in $H^1(\mathrm{Diff}_{\mathcal{S}}(M)_0, C^\infty(M))$ is an invariant of the conformal structure $\mathcal{S}$.

We see here that the global extended Lee homomorphism, which is a conformal invariant as well, is the sum of this invariant and another piece, which is not a conformal invariant.

4.1. The Calabi homomorphism

Proposition 3 suggests the following proposition.

Proposition 5. *The mapping*

$$[\phi] \mapsto \int_0^1 \lambda\pi^*(\phi_t^* \mathrm{i}(\dot{\phi_t})\Omega)\,\mathrm{d}t$$

is a well-defined surjective homomorphism on the universal cover $\widetilde{\mathrm{Ker}}\,\mathcal{L}$ of $\mathrm{Ker}\,\mathcal{L}$

$$\tilde{S} : \widetilde{\mathrm{Ker}}\,\mathcal{L} \to H^1_{c\mathcal{A}}(\tilde{M}).$$

Remarks.

1. In Theorem 2, to define a lifting $\tilde{\phi}$ of $\phi \in \mathrm{Diff}_{\mathcal{S}}(M)$, we do not need an isotopy ϕ_t from id to ϕ in $\mathrm{Diff}_{\mathcal{S}}(M)$. Any homotopy in the group of homeomorphism just does the job, unlike in [10]. Hence, our construction is valid for a group larger than the identity component $\mathrm{Diff}_{\mathcal{S}}(M)_0$.
2. A fragmentation lemma for elements in $\mathrm{Ker}\,\tilde{S}$ can also be proved.

5. Applications to mechanics

5.1. Twisted Hamiltonians [14]

Symplectic geometry is the natural setting for classical mechanics. Given a function H on a symplectic manifold (M, Ω), we use the symplectic form Ω to define the Hamiltonian vector field X_H: the unique vector field X_H such that $\Omega(X_H, \xi) = \mathrm{d}H(\xi)$ for any vector field ξ. Hamilton equations then take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = X_H(\gamma_t).$$

If we only have an LCS pair (Ω, ω) on M, we still can define Hamiltonian vector fields: just take functions H_i defined on each open set U_i of an open cover as above which satisfy the transition rule: $H_i = c_{ij}H_j$ on $U_i \cap U_j$, and define local Hamiltonian vector fields X_{H_i} in each open set by $\Omega_i'(X_{H_i}, \xi) = \mathrm{d}H_i(\xi)$. Here $\Omega_j = \Omega|_{U_j}$ and $\Omega_j' = \lambda_j \Omega_j$, and

$\Omega'_i = c_{ij}\Omega'_j$ (see notations in Section 3). In $U_i \cap U_j$, we have $\Omega'_i(X_{H_i}, \cdot) = \mathrm{d}H_i$, which rewrite as

$$c_{ij}\Omega'_j(X_{H_i}, \cdot) = \Omega'_i(X_{H_i}, \cdot) = \mathrm{d}H_i = \mathrm{d}(c_{ij}H_j) = c_{ij}\,\mathrm{d}H_j = c_{ij}\Omega'_j(X_{H_j}, \cdot).$$

Simplifying by the non-zero constant c_{ij}, we get $\Omega'_j(X_{H_i}, \cdot) = \Omega'_j(X_{H_j}, \cdot)$. This says that X_{H_i} and X_{H_j} agree on the intersection $U_i \cap U_j$. Hence, we have Hamilton equations defined globally, and automorphisms of the Γ-structure defined by the LCS structure, i.e. locally conformal diffeomorphisms respect the local Hamilton equations, and hence the global one.

Therefore, LCS geometry is a more general setting for Hamiltonian mechanics than symplectic geometry. The set (H_i) of functions satisfying the condition $H_i = c_{ij}H_j$ are sections of some line bundle over M, and are called “twisted” Hamiltonians. This motivation of LCS geometry was given by Vaisman [14].

5.2. *Mechanics on the regular cover*

We are now going to formulate Hamilton equations on the regular cover $\tilde{M}$ associated with the Lee form on an LCS manifold M. Let H be a smooth function on M. Consider the unique vector field X_H on M defined by the following equation:

$$\mathrm{i}(X_H)\Omega = \mathrm{d}H + H\omega.$$

A short calculation shows that

$$L_{X_H}\Omega = (-\mathrm{i}(X_H)\omega)\Omega,$$

i.e. X_H is an automorphism of the locally conformal structure, which is in the kernel of the extended Lee homomorphism l. Therefore, any lift Y_H of X_H is a symplectic vector field of $(\tilde{M}, \Omega')$. Note that

$$\mathrm{i}(Y_H)\Omega' = \lambda\pi^*(\mathrm{i}(X_H)\Omega) = \lambda\pi^*(\mathrm{d}H + H\omega) = (\mathrm{d}H')\lambda + H'\lambda\,\mathrm{d}\ln\lambda = \mathrm{d}(\lambda H'),$$

where $H' = H \circ \pi$. Hence Y_H is the Hamiltonian vector field on $(\tilde{M}, \Omega')$, with Hamiltonian $\lambda H'$. The Hamilton equations on $(\tilde{M}, \Omega')$ can then be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = Y_H(\gamma_t)$$

on $(\tilde{M}, \Omega')$.

For $f, g \in C^\infty(M)$, the function $(\Omega'(Y_f, Y_g))/\lambda$ is a basic function which is equal to the Jacobi bracket defined below. This globalizes a formula of Guerida and Lichnerowicz [9].

If (Ω, ω) is an LCS pair on a smooth manifold M, then $\tilde{\Omega} : T(M) \to T^*(M),\ X \mapsto \mathrm{i}(X)\Omega$ is an isomorphism.

Like in the symplectic case, we get a non-degenerate skew symmetric tensor field P corresponding to $\tilde{P} = \tilde{\Omega}^{-1} : T^*(M) \to T(M)$. The condition $\mathrm{d}\Omega = -\omega \wedge \Omega$ translates as

$$[[P, P]] = 2E \wedge P \quad \text{and} \quad [[E, P]] = 0,$$

where $E = P(\omega)$. Here $[[\cdot]]$ is the Schouten-Nijenhuis bracket, defined on $\Lambda^p(TM)$ as follows: $[[X, Y]] = [X, Y]$ the Lie bracket if X, Y are vector fields. If X is a vector field and $Y = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_k$, then $[[X, Y]] = \sum(-1)^j Y_1 \wedge \cdots \wedge [X, Y_j] \wedge Y_{j+1} \wedge \cdots \wedge Y_k$. It is then extended bilinearly.

A couple (E, P) satisfying the identities as above is called a *Jacobi structure*. Therefore, an LCS structure gives raise to a Jacobi structure (P, E) on M [9]. The (Jacobi) bracket $\{f, g\}$ of two functions on M is the function

$$\{f, g\} = P(\mathrm{d}f) \cdot g + f\,\mathrm{d}g(E) - g\,\mathrm{d}f(E).$$

With the Jacobi bracket, the space $C^\infty(M)$ of smooth functions on M becomes a Lie algebra. We have

$$\{f, g\} = \frac{\Omega'(Y_f, Y_g)}{\lambda}.$$

The local version of this formula can be found in [9], where it is proved that LCS manifolds are the same thing as even-dimensional transitive Jacobi manifolds.

For more comments on the connection between other structures underlying mechanics, see [5].

Acknowledgements

I would like to thank S. Haller for helpful comments.

References

[1] A. Banyaga, Sur la structure des diffeomorphismes qui preservent une forme symplectique, Comment. Math. Helv. 53 (1978) 174–227.
[2] A. Banyaga, Invariants of contact structures and transversally oriented foliations, Ann. Global Anal. Geom. 14 (1996) 427–441.
[3] A. Banyaga, The structure of classical diffeomorphism groups, Mathematics and its Applications, Vol. 400, Kluwer Academic Publisher, Dordrecht, the Netherlands, 1997.
[4] A. Banyaga, On essential conformal groups and a conformal invariant, J. Geom. 68 (2000) 10–15.
[5] A. Banyaga, Some geometrical structures underlying mechanics, in: J. Govaerts, M.N. Hounkonnou, W.A. Lester Jr. (Eds.), Contemporary Problems in Mathematical Physics, World Scientific, Singapore, 2000, pp. 109–117.
[6] E. Calabi, On the group of automorphism of a symplectic manifold, in: Problems in Analysis, Symposium in Honour of S. Bochner, Princeton University Press, Princeton, NJ, 1970, pp. 1–26.
[7] Y. Eliashberg, M. Gromov, Convex symplectic manifolds, in: Proceedings of the 52nd Symposia on Pure Mathematics, Part II, American Mathematical Society, Providence, RI, 1991, pp. 135–162.
[8] C. Godbillon, Elements de topologie algebriques, Hermann, Paris, 1971.
[9] F. Guerida, A. Lichnerowicz, Geometrie des algebres de Lie locales de Kirillov, J. Math. Pure Appl. 63 (1984) 407–484.
[10] S. Haller, T. Rybicki, On the group of diffeomorphisms preserving a locally conformal symplectic structure, Ann. Global Anal. Geom. 17 (1999) 475–502.
[11] P. Iglesias, Arithmetique des rapports de similitudes symplectiques, Comp. Math. 95 (1995) 235–245.
[12] S. Kobayashi, Transformation groups in differential geometry, Erg. Math. Grenzgeb, Vol. 70, Springer, Berlin, 1972.
[13] H.C. Lee, A kind of even-dimensional differential geometry and its application to exterior calculus, Am. J. Math. 65 (1943) 433–438.
[14] I. Vaisman, Locally conformal symplectic manifolds, Int. J. Math. Math. Sci. 8 (3) (1983) 521–536.